THE TRANSVERSE RESISTIVE WALL INSTABILITIES IN THE NAL MAIN RING

A. G. Ruggiero

October 25, 1970

I. INTRODUCTION

The transverse coherent instability of multibunched beams has already been investigated by Courant and Sessler[1] (CS) in the special case of dipolar coherent oscillations and of resistive vacuum-chamber wall.

The purpose of the present paper is to apply and to extend the results of the CS investigation to the NAL main ring.

The first of our goals is to identify the unstable modes of a beam circulating in the NAL main ring for the special case of zero betatron frequency spread. We shall deal with the special case of full beam in the main ring composed of $M = 1113$ bunches equally spaced and with equal particle numbers.

The second step of our investigation is the computation of the unstable modes growth time, still in absence of betatron frequency spread.

Computation shows that almost all the unstable modes exhibit a growth time much less, or at least comparable to the acceleration period of 1.6 seconds in the main ring. Thus, almost all the unstable modes are dangerous too.

The last of our goals is the computation of the minimum betatron spread required to reduce to zero the growth time of the most unstable mode, i.e., of that mode with the smallest growth time in absence of spread.

Our computation, which is based on the exact solution of the dispersion relation, shows that the required frequency spread is much too sensitive to the tail length of the particle distribution, at least for the special case of the NAL main ring, so that it can be a problem to make a choice for the "minimum" required spread as we have not enough information about the particle distribution. Besides, the spread is depending on the (complex) beam-wall coupling factor with the effect to increase its uncertainty because of the uncertainty by which we can only compute the coupling factor.

II. MAIN NOTATION

R = radius of the bunch orbit

L = bunch total length

a = beam radius

b = vacuum-chamber inner radius

e = particle charge

m_0 = particle rest mass

c = light velocity

ρ = particle velocity to light velocity ratio

γ = particle total energy to rest energy ratio

N = number of particles per bunch

ω_0 = angular revolution frequency of the beam

σ = conductivity of the wall material

ν_c = nominal betatron oscillations number per turn

III. BASIC THEORETICAL RESULTS

The main result from the CS paper is a dispersion relation that can be written in the following way

$$\lambda_m I = 1, \tag{1}$$

where I is the following dispersion integral

$$I = \int \frac{f(\nu_s) d\nu_s}{\nu_s^2 - \nu_m^2} . \tag{2}$$

$f(\nu_s)$ is the particle distribution function in the betatron wavelengths number per turn ν_s. This function is normalized in the following way

$$\int f(\nu_s)\, d\nu_s = 1.$$

ν_m appearing at the denominator of the integral (2) is the (complex) collective betatron wavelengths number per turn. λ_m is one of the M bunch eigenvalues.

As we are only interested in the special case of bunches with the same number of particles N and equally spaced, we show below the expression for λ_m for this special case as derived from the CS paper.

If the collective bunches oscillations are taken with the form

$$\exp(i\nu_m \omega_0 t), \tag{3}$$

we have

$$\lambda_m = \frac{N}{m_0 \gamma \omega_0^2} \left\{ U' + W' \left[\sqrt{M}\, G\left(2\pi \, \frac{m + \nu_m}{M} \right) - i\nu_m \frac{8}{15} \sqrt{\frac{L}{2R}} \right] \right\}, \quad (4)$$

where

$$U' = (e^2 U/L) + \frac{8}{3} W' \sqrt{\frac{2R}{\pi L}} \quad (5)$$

$$W' = e^2 W/2\pi R\sqrt{\omega_0}\,, \quad (6)$$

and for circular geometry

$$U = -\frac{2}{\gamma^2} \left(\frac{1}{a^2} - \frac{1}{b^2} \right) \quad (7)$$

$$W = (2c\beta^2/b^3)/\sqrt{\pi\sigma}\,. \quad (8)$$

The bunching function $G(2\pi, x)$ has been investigated in more details in the CS paper. It can be defined by the following series:

$$G(2\pi, x) = \sqrt{2}\sum_{k=1}^{\infty} \frac{e^{-2\pi i k x}}{\sqrt{k}}\,.$$

Here we want to mention a fundamental property of this function. It is periodic in x and its imaginary part is <u>negative</u> when x lies in the first half of one period (for example, between 0 and 1/2), and <u>positive</u> when x lies in the second half (for example, between 1/2 and 1).

The third term, in ν_m, at the right-hand side of Eq. (4) is the contribution of the <u>internal fields</u> that one particle suffers by effect of the other particles in the same bunch. This term is usually very small.

As we expect ν_m to be not so much different from ν_c, the ν_s value of the particle distribution center, we replace, in the following, ν_m by ν_c at the right-hand side of Eq. (4).

In the following we shall make use also of the notation

$$\nu_m = \nu_c + \Delta_m,$$

where Δ_m is the complex betatron wavelength number shift. We can split Δ_m in its real and imaginary part, respectively Δ_{r_m} and Δ_{i_m}, and definitely assume

$$|\Delta_m| << \nu_c.$$

From Eq. (3) we see that the stability condition is

$$\Delta_{i_m} > 0. \tag{9}$$

A simple case to solve is the following

$$f(\nu_s) = \delta(\nu_s - \nu_c), \tag{10}$$

where δ (x) is the Dirac function.

Insertion of Eq. (10) in Eq. (2) gives

$$I = \frac{1}{\nu_c^2 - \nu_m^2},$$

and if $|\lambda_m| << \nu_c^2$,

$$\nu_m = \pm\nu_c \mp \frac{\lambda_m}{2\nu_c}. \tag{11}$$

As we want ν_m to go to ν_c when λ_m moves to zero, we keep the upper sign at the right-hand side of Eq. (11), and we eliminate the solution with the lower sign. Thus we have

$$\Delta_m = \frac{\lambda_m}{2\nu_c}, \tag{12}$$

and the stability condition is from Eq. (9),

$$\lambda_{i_m} = \text{imaginary part of } \lambda_m < 0. \tag{13}$$

Thus if we neglect the internal forces term, and we consider only the case with $M >> \nu_c$, we have, from the theorem about the sign of $G(2\pi, x)$, that the modes with m satisfying the condition

$$0.5 < \frac{m + \nu_c}{M} < 1.0 \tag{14}$$

are unstable, and all the others with m outside the range Eq. (14) are stable.

The growth time, in absence of ν_s spread, of the unstable mode can be derived from Eq. (12). We have

$$\tau_m = \frac{2\nu_c}{\omega_0 \lambda_{i_m}}. \tag{15}$$

IV. APPLICATION TO THE NAL MAIN RING

We have

$$\nu_c = 20.25$$

$$M = 1113,$$

and we derive the unstable modes from Eq. (14). They are

$$537 \le m \le 1092.$$

If we take into account the imaginary quantity in ν_m at the right-hand side of Eq. (4), only a few modes close to m = 537 are converted in stable modes.

In the following we split λ_m in its real, λ_{r_m}, and imaginary, λ_{i_m}, part. We want to compute λ_{i_m}, λ_{r_m}, and the growth time τ_m, in absence of betatron spread, for the unstable modes in the NAL main ring. For this purpose we take the following numbers:

$$\begin{aligned}
\gamma &= 10.0 \\
\beta &= 1.0 \\
a &= 0.5 \text{ cm} \\
b &= 2.0 \text{ cm} \\
\sigma &= 10^{17} \text{ s}^{-1} \\
L &= 100.0 \text{ cm} \\
R &= 1000.0 \text{ m} \\
N &= 4.2 \times 10^{10} \\
M &= 1113 \\
\nu_c &= 20.25.
\end{aligned} \tag{16}$$

The results are shown in Table I below.

λ_{i_m} increases and λ_{r_m} and τ_m decrease steadily when the mode number m runs from 539 up to 1092.

The first two modes, m = 537 and 538, are stable when the internal forces are taken into account. The modes m ≥ 539 are unstable even taking them into account.

Table I. Coupling Factors and Growth Time Vs Mode Number.

m	λ_{r_m}	λ_{i_m}	τ_m
537, 538	stable, taking into account internal forces		
539	-4.73	1.58×10^{-5}	8.5 sec
540	-4.73	4.35×10^{-5}	3.1 sec
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.
1091	-4.60	0.150	0.9 msec
1092	-4.54	0.229	0.6 msec

Practically all the modes are dangerous because except for the very first few at the low limit, they exhibit a growth time τ_m much less, or less, or comparable to the acceleration period which is of 1.6 seconds.

The most dangerous mode is m = 1092 with a growth time of about half a millisecond. It should be remembered that the designed synchrotron oscillation period in the NAL main ring is about 10 milliseconds.

Thus only a few of the modes, and the higher ones, have a growth time smaller than the synchrotron oscillation period, T. The others exhibit a growth time either comparable to T or much longer than T, and hence, the CS theory used here cannot be rigorously supposed valid for them, as the particle motion within the bunches has been neglected.

It is soon seen from the Table I as λ_{r_m} changes weakly, whereas λ_{i_m} is well sensitive to the mode number m. Besides, the ratio $\lambda_{i_m} / |\lambda_{r_m}|$ is very small for any of the unstable mode and has a maximum of about 0.05 at the mode m = 1092.

We want now to discuss the computation accuracy of λ_{r_m}, λ_{i_m}, and hence, of τ_m.

The dispersion relation Eq. (1) relates together the real and the imaginary betatron shifts, respectively Δ_{r_m} and Δ_{i_m}, the nominal betatron number ν_c, the real and the imaginary form factors, respectively, λ_{r_m} and λ_{i_m}, and two quantities, δ and η, characteristic of the particle distribution function $f(\nu_s)$. δ and η will be described in the next sections; here, it is sufficient to say that δ is a measure of the function width, and hence, of the betatron spread within the beam, and η is a measure of the particle concentration around the center $\nu_s = \nu_c$, and hence, of the distribution tail length.

The dispersion relation Eq. (1) takes then a general form like the following:

$$H(\lambda_{r_m}, \lambda_{i_m}, \nu_c, \Delta_{r_m}, \Delta_{i_m}, \delta, \eta) = 0. \tag{17}$$

A special case that will be discussed deeply in the next section is $\Delta_{i_m} \to 0^-$, which is told also "at the limit of stability." In this case, Δ_{i_m} is dropped at the left hand side of Eq. (17), and we should be able to solve for the minimum spread $\delta_{\Delta_{i_m} \to 0^-} = \delta_m$, obtaining

$$\delta_m = K(\lambda_{r_m}, \lambda_{i_m}, \nu_c, \eta). \qquad (18)$$

Since Eq. (17) is a complex relation, it can be split in two real relations, one of which can be used to solve for Δ_{r_m}. This is the reason for which Δ_{r_m} also has been dropped in Eq. (18).

Thus, keeping ν_c and η fixed, the spread δ_m is related to λ_{r_m} and λ_{i_m}. Any inaccuracy by which we know λ_{r_m} and λ_{i_m} is unavoidably transferred in an inaccuracy for δ_m.

We calculated λ_{r_m} and λ_{i_m} by using the ten parameters listed in Eq. (16). The last four (R, M, N, and ν_c) are sure parameters; we know them with high accuracy. We did not observe any change of λ_{r_m} and λ_{i_m} listed in the Table I when we moved ν_c down and up around 20.25. γ and β also are sure parameters, but they change with the time. Nevertheless, the beam in the main ring is always so relativistic that our assumption $\beta = 1$ should not affect our results. γ changes from 10 at the injection up to 210 at the acceleration top. We referred to $\gamma = 10$ to have the maxima of λ_{r_m} and λ_{i_m}, as we know from Eq. (4), that λ_{r_m} goes like γ^{-3} and λ_{i_m} like γ^{-1}. If we fix our attention for the moment to the mode m = 1092, we have that $\lambda_{i_m} / |\lambda_{r_m}|$ is 0.05 at $\gamma = 10$, 1 at $\gamma = 45$, and 22 at the top $\gamma = 210$.

The remaining four parameters (a, b, L, and σ) are not sure parameters. We believe that the cylindric model can well be applied to the NAL main-ring case provided the equivalent radii a and b can be guessed. We observe that λ_{i_m} is sensitive only on b and not on a. Unfortunately, the dependence on b is of cubic power, so that if we double b, λ_{i_m} is affected by a factor of 8 (!).

λ_{r_m} is sensitive on both a and b, as one can see from Eq. (7), but as generally a is much smaller than b, we can expect a stronger dependence on a than on b.

Also, L and σ can be only inaccurately guessed. The vacuum-chamber steel has a well known conductivity σ_{st}, but other conductors and insulators are located around the ring circumference resulting in an averaged conductivity σ that we can hardly measure. The bunch length L is a function of the time which is still to be fixed.

Fortunately, λ_{i_m} has no L-dependence and is weakly sensitive on σ, as only the square root enters the denominator of its expression [see Eqs. (4) to (8)].

Thus, since it depends only on λ_{i_m} [see Eq. (15)], the growth time τ_m depends only on γ, σ, and b according to

$$\tau_m \sim \text{constant x} (\gamma b^3 \sqrt{\sigma}),$$

where a and L do not appear.

λ_{r_m} can be split in three terms as it follows,

$$\lambda_{r_m} \sim -\frac{c_1}{\gamma^3 L} + \frac{c_2}{\gamma\sqrt{\sigma L}} + \frac{c_3}{\gamma\sqrt{\sigma}}\ , \qquad (19)$$

where c_2 and c_3 have a b^{-3} dependence, and c_1 a combined dependence on a and b as given by Eq. (7).

For perfectly conductive material ($\sigma \to \infty$), the second and third terms drop and λ_{r_m} is given only by the first term. For L = 100 cm and $\sigma = 10^{17}\ s^{-1}$, the last two terms contribute in the total only in the measure of 5%. In order to change appreciably λ_{r_m}, σ should go down considerably. For instance, λ_{r_m} is halved at $\sigma = 10^{15}\ s^{-1}$. Thus we can neglect the last two terms at the right-hand side of Eq. (19) for low γ values, although they can contribute appreciably at higher energy.

In the following it can be useful to refer to the coupling factor α_m per particle defined as follows:

$$\alpha_m = \lambda_m / N. \tag{20}$$

Obviously α_m can be split in its real and imaginary parts

$$\alpha_m = \alpha_{r_m} + i\, \alpha_{i_m} .$$

It is soon seen from Eq. (4) that α_{r_m} and α_{i_m} are not depending on N.

V. THE DISPERSION RELATION

We state the following:

a. $f(\nu_s)$ is zero anywhere except in a region around $\nu_s = \nu_c$ between $\nu_c - \zeta$ and $\nu_c + \zeta$. It is symmetric around $\nu_s = \nu_c$.

b. The total spread, $2\,\delta$, of the distribution is defined by taking the total width of $f(\nu_s)$ at half of its maximum.

c. δ is a very small quantity with respect to ν_c.

d. The collective betatron number ν_m can be written in the following way

$$\nu_m = \nu_c + \Delta_m,$$

where Δ_m is a complex quantity whose real part Δ_{r_m} is supposed to be much smaller than ν_c. Δ_{r_m} is expected to have the same order of magnitude of δ or larger.

e. The imaginary part, Δ_{i_m}, of the ν-shift Δ_m can take, in principle, any value. Nevertheless, since we want to calculate the minimum spread $\delta = \delta_m$ for the instability compensation at the limit of stability, we take Δ_{i_m} as a very small and negative quantity. The dispersion integral I can be easily split in two integrals

$$I = \frac{1}{2\nu_m} \int \frac{f(\nu_s)}{\nu_s - \nu_m} \, d\nu_s - \frac{1}{2\nu_m} \int \frac{f(\nu_s)}{\nu_s + \nu_m} \, d\nu_m .$$

For the assumptions stated above, the second integral can be neglected as it contributes only weakly to I.

Thus, we have from Eq. (1)

$$\frac{\lambda_m}{2\nu_c} \int_{\nu_c - \zeta}^{\nu_c + \zeta} \frac{f(\nu_s)}{\nu_s - \nu_m} \, d\nu_s = 1 .$$

We replaced ν_m by ν_c in the factor outside the integral. That is certainly a good approximation for the statements d and e. We shall replace ν_m by ν_c also in the expression for λ_m.

Changing the integral variable with

$$x_s = \frac{\nu_s - \nu_c}{\delta}$$

and setting

$$x_m = \frac{\Delta_m}{\delta},$$

we have

$$\Lambda_m \int_{-\epsilon}^{+\epsilon} \frac{g(x_s)}{x_s - x_m}\, dx_s = 1 \tag{21}$$

with

$$\Lambda_m = \frac{\lambda_m}{2\nu_c \delta}, \tag{22}$$

and $g(x) = \delta\, f(\nu_s)$ is the new scaled distribution function normalized to unit. It is required that the function $g(x)$ is chosen in such a way that

$$g(\pm 1) = \frac{1}{2}\, g(0).$$

We split x_m and Λ_m in their real and imaginary parts

$$x_m = x_{r_m} + i x_{i_m}$$

$$\Lambda_m = \Lambda_{r_m} + i\Lambda_{i_m},$$

and we operate the limit

$$x_{i_m} \to 0^-,$$

so that we obtain from Eq. (21)

$$(\Lambda_{r_m} + i\ \Lambda_{i_m})\ (F_{r_m} + i\ F_{i_m}) = 1, \tag{23}$$

where

$$F_{r_m} = \int_{-\epsilon}^{+\epsilon} \frac{g(x_s)}{x_s - x_{r_m}}\, dx_s, \tag{24}$$

and

$$\begin{aligned} F_{i_m} &= -\pi\, g\,(x_{r_m}), \text{ for } |x_{r_m}| < \epsilon \\ &= 0, \qquad\qquad \text{ for } |x_{r_m}| > \epsilon . \end{aligned} \tag{25}$$

We observe soon that F_{r_m} and F_{i_m} are only functions of the running parameter x_{r_m} (and of the integral limit ϵ, but this is only an artificial parameter). All the other quantities, ν_c, the minimum spread δ_m for compensation and the coupling factor λ_m, enter Λ_{r_m} and Λ_{i_m}.

From Eq. (23) we have, at last,

$$\Lambda_{r_m} + i\,\Lambda_{i_m} = \frac{F_{r_m} - i\,F_{i_m}}{F_{r_m}^{\ 2} + F_{i_m}^{\ 2}} . \tag{26}$$

The above equation is the conformal mapping of the curve $x_{i_m} = \alpha$ in the plane (x_{r_m}, x_{i_m}) to the plane $(\Lambda_{r_m}, \Lambda_{i_m})$ for the special case $\alpha \to 0^-$. That results in a curve Γ in the plane $(\Lambda_{r_m}, \Lambda_{i_m})$ with x_{r_m} as running parameter. The curve Γ bounds a region which can be called the "stable region". In fact, all the experimental points of coordinates Λ_{r_m} and Λ_{i_m} falling inside represent stable beam oscillations. The points on the curve show the limit of stability.

The mapping depends upon the choice of the distribution function g (x) that, for this reason, can be called the "generatrix function."

In the next section we draw down conformal mapping Eq. (26) for several generatrix functions, g (x).

Here, we conclude by observing that the half-plane $\Lambda_{i_m} < 0$ is certainly a stable region as any point inside has $\lambda_{i_m} < 0$ which corresponds to the stability condition. Besides, we see from Eq. (25) and Eq. (26) that Λ_{i_m} is a positive quantity or zero, whereas Λ_{r_m} has the opposite sign of $-x_m$. Thus the boundary curve Γ is symmetric with respect to the axis $\Lambda_{r_m} = 0$ and lies surely in the upper half-plane, corresponding to $\Lambda_{i_m} > 0$, with the effect to widen up the stable region.

VI. THE EFFECT OF THE DISTRIBUTION TAILS

We want here to investigate the effect of the particle distribution tails.

For this purpose we shall make use of several "generatrix function" g (x) in the integral (21). A simple way to measure the tail length of the distribution is to calculate the quantity

$$\eta = \int_{-1}^{+1} g(x)\, dx ,$$

which gives the relative number of particles inside the width of the distribution. When η approaches 1, the tails vanish, and they appear and stretch as η decreases to zero.

As g (x) is normalized to unit and symmetric to x, we cannot expect η less than 0.5, corresponding to a Lorentz distribution.

The "generatrix functions" g (x), used in our investigation are tabulated in Table II.

We prepared a computer program for the numerical calculation of the Cauchy principal value of the integral (24), and we drew down the conformal mapping Eq. (25) for each of the functions in Table II. The results of the mapping are shown in Fig. 1.

The straight line parallel to the Λ_{r_m} axis and crossing the Λ_{i_m} axis at $\Lambda_{i_m} = 1$ is the boundary curve Γ for a Lorentz function which has the smallest value of η (=0.5). The next curve met moving down corresponds to the gauss function with a higher η (=0.76). The most inside curve, which hence bounds the less wide stable region, corresponds to the second order parabola function with the largest η (=0.886) we considered. Each curve is marked by an integer number for identification. These numbers are listed in the last column of Table II.

One important result is the following. The distribution with larger tails yields to wider stable region containing entirely any other stable region corresponding to distributions with shorter tails.

All the functions in Table II have the property to be continuous at any x. We considered also the rectangular distribution function which does not hold this property and thus cannot be rigorously inserted in the group.

The dashed curve in Fig. 1 corresponds to the rectangular function. The fact that now the curve is closed at the origin of the $(\Lambda_{r_m}, \Lambda_{i_m})$ plane is to be indebted just to the discontinuities at the two sides of the function.

If we know the coupling factor per unit length α_m, Eq. (20), the number of particles within a bunch, N, and the ν spread, δ, within the beam, we can calculate Λ_{r_m} and Λ_{i_m} for a practice case. We could, then, observe whether the point of coordinate $(\Lambda_{r_m}, \Lambda_{i_m})$ is below or above one of the boundary curves in Fig. 1, and thus state if the beam is stable or unstable.

We wish to suggest a different method. We rewrite Λ_m, Eq. (22), in the following way:

$$\Lambda_m = \frac{\alpha_m}{2\nu_c} \Big/ (\delta_m/N),$$

where we introduced the minimum spread per particle, δ_m/N, required for the beam compensation. We want to compute δ_m/N. For this purpose we observe that the ratio $\Lambda_{r_m}/\Lambda_{i_m}$ is constant for a practice case and is given by

$$\frac{\Lambda_{r_m}}{\Lambda_{i_m}} = \frac{\alpha_{r_m}}{\alpha_{i_m}} = \frac{\lambda_{r_m}}{\lambda_{i_m}} . \tag{27}$$

Then we show in Fig. 1 a straight line passing the origin and with angular coefficient equal to the ratio (27).

The running parameter over the straight line is now δ_m/N. Changing δ_m/N we move simply the experimental point along the straight line.

The coordinates of the point at the intersection between the experimental straight line and one boundary curve Γ can be used to calculate δ_m/N if we know α_m.

We used this method for the computation of δ_m/N in the NAL main ring. The results are shown in Fig. 2. We have η in the abscissa, and the minimum half spread per particle, normalized to $\alpha_{i_m}/2\nu_c$, in the ordinate. We see three curves corresponding to different ratios $\alpha_{i_m}/|\alpha_{r_m}|$. The computed points are shown by the thick points. The plot is universal and can be applied to any practice case.

For the special case of the NAL main ring we take

$$\alpha_{i_m}/|\alpha_{r_m}| = 0.05,$$

and thus we see that δ_m/N should range in the following interval

$$\frac{\alpha_{i_m}}{2\nu_c} < \frac{\delta_m}{N} < 70\,\frac{\alpha_{i_m}}{2\nu_c}\,,$$

as η changes from 0.5 to 1.

This gives, for the designed number of particles per bunch $N = 4.2 \times 10^{10}$, the minimum total spread

$$0.01 < 2\delta_m < 0.8$$

which is really a large range!

But if we take into consideration a Gaussian particle distribution (which is not at all unphysical), we would obtain

$$2\delta_m \sim 0.1 \quad .$$

The same operation for the computation of δ_m/N can be used to determine the real betatron shift x_{r_m} . The results are shown in Fig. 3. Also, here we have three curves corresponding to different ratios $\alpha_{i_m}/|\alpha_{r_m}|$.

We observe that for distributions with long tails, it results in a real betatron shift Δ_{r_m} of several times the half-spread δ_m.

REFERENCE

[1] E. D. Courant, A. M. Sessler, Rev. Sci. Instr. 37, 1579 (1966).

Table II. Distribution Functions and Their Tail Coefficients.

Name	g(x)	ϵ	η	Curve No.
Lorentz	$1/\pi(1+x^2)$	∞	0.500	1
Gauss	$\sqrt{(\lg 2)/\pi}\,\exp(-x^2 \lg 2)$	∞	0.76	2
Parabola n = 5	$A_n\left(1-\alpha_n^2 x^2\right)^{n\ a}$	$1/\alpha_n$ [a]	0.795	3
Parabola n = 4	$A_n\left(1-\alpha_n^2 x^2\right)^{n\ a}$	$1/\alpha_n$ [a]	0.802	4
Parabola n = 3	$A_n\left(1-\alpha_n^2 x^2\right)^{n\ a}$	$1/\alpha_n$ [a]	0.812	5
Squared cosine	$\frac{1}{2}\cos^2(\pi x/4)$	2	0.818	6
Parabola n = 2	------	$1/\alpha_2$ [a]	0.835	7
Truncated cosine	$\frac{\pi}{6}\cos(\pi x/3)$	3/2	0.866	8
Parabola n = 1	------	$1/\alpha_1$ [a]	0.886	9

[a] For the parabola functions it is

$$A_n = \alpha_n \frac{(2n+1)!}{2^{2n+1}\,(n!)^2}$$

$$\alpha_n = \sqrt{1 - \frac{1}{n\sqrt{2}}}$$

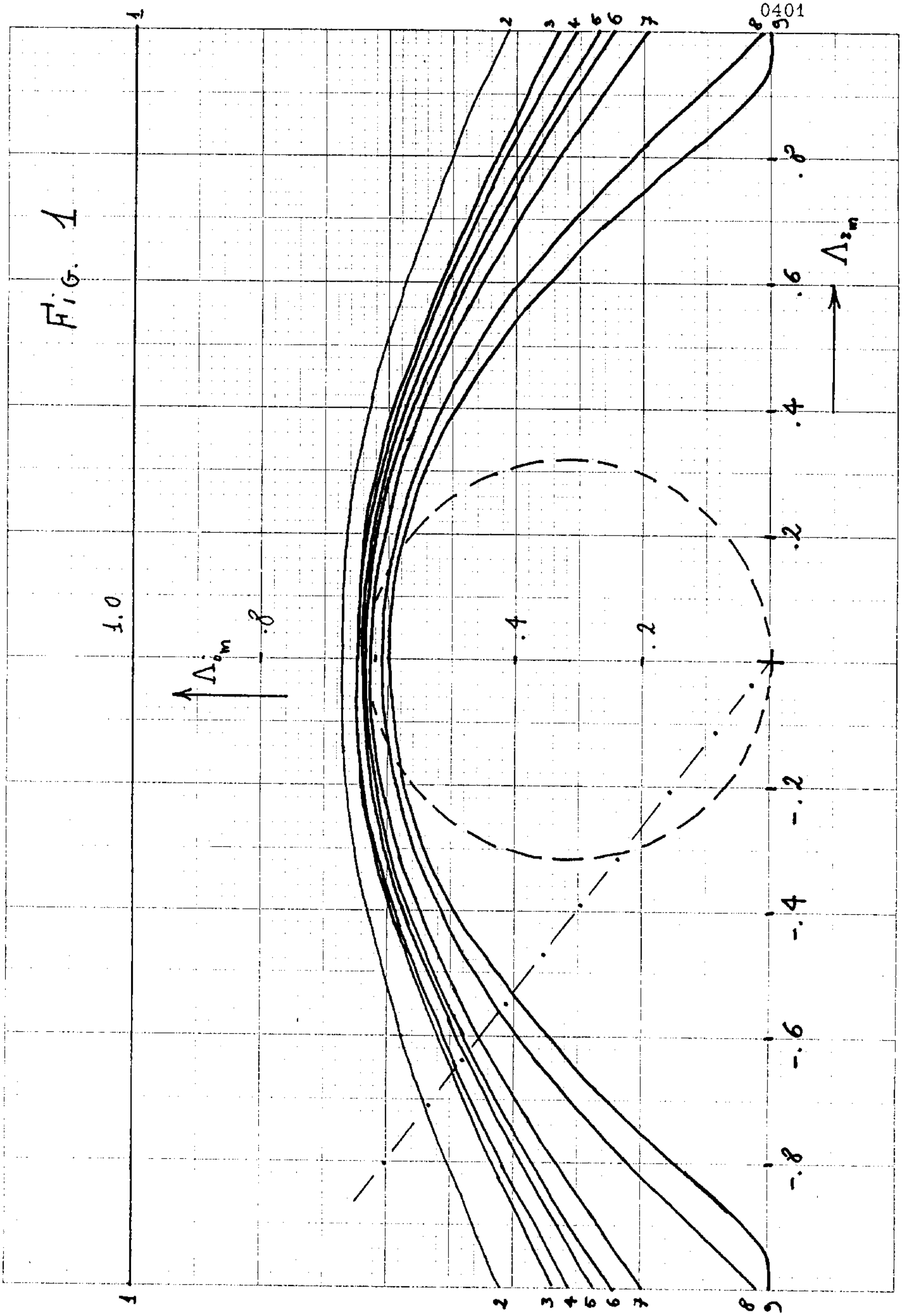
Fig. 1
1.0
.8
.4
.2
Λ_{im}
Λ_{rm}
-.8
-.6
-.4
-.2
.2
.4
.6
.8
1
2
3
4
5
6
7
8
9

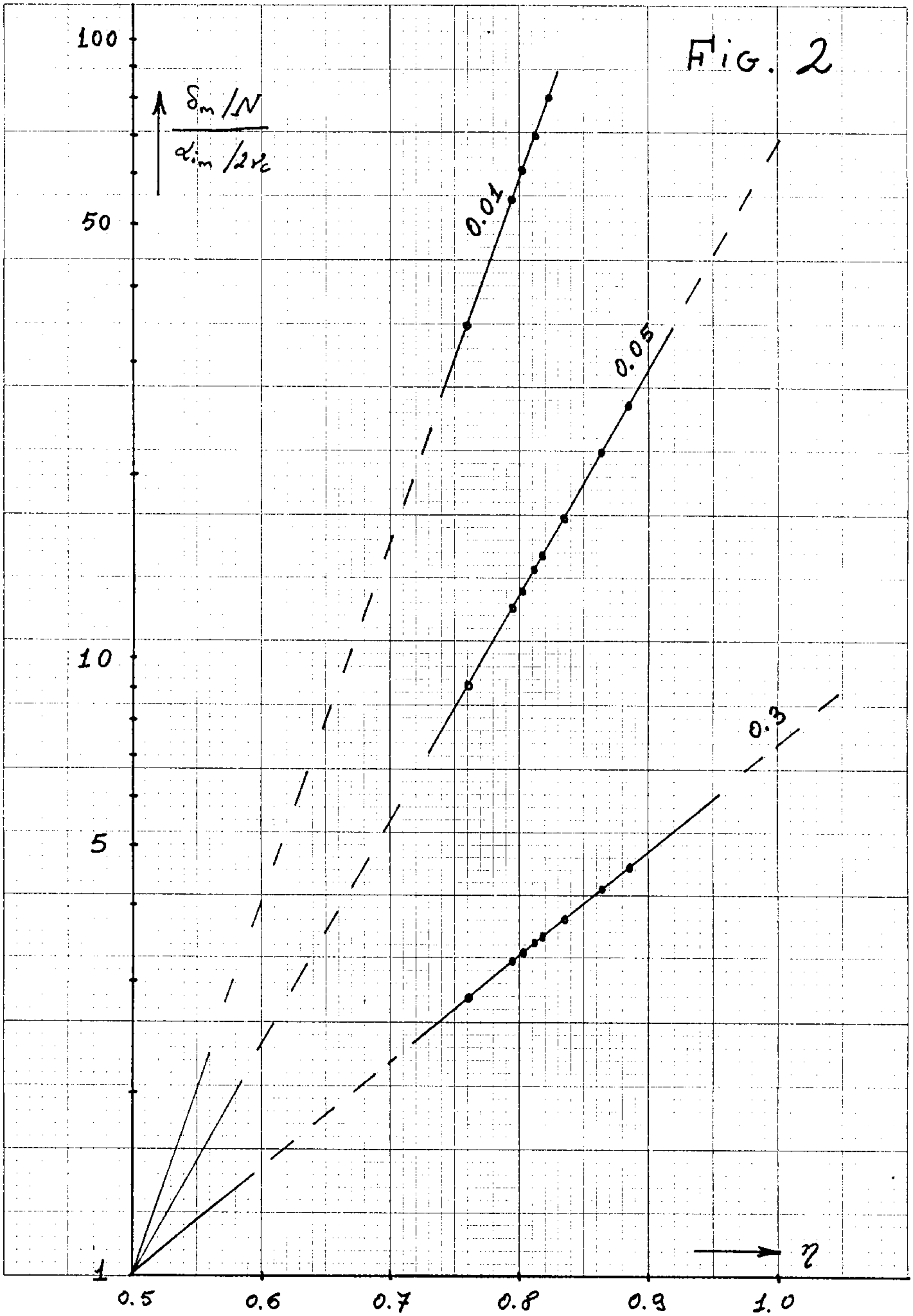
FIG. 2
δ_m / N / $\alpha_{im} / 2\nu_c$
0.01
0.05
0.3
100
50
10
5
1
0.5
0.6
0.7
0.8
0.9
1.0
η

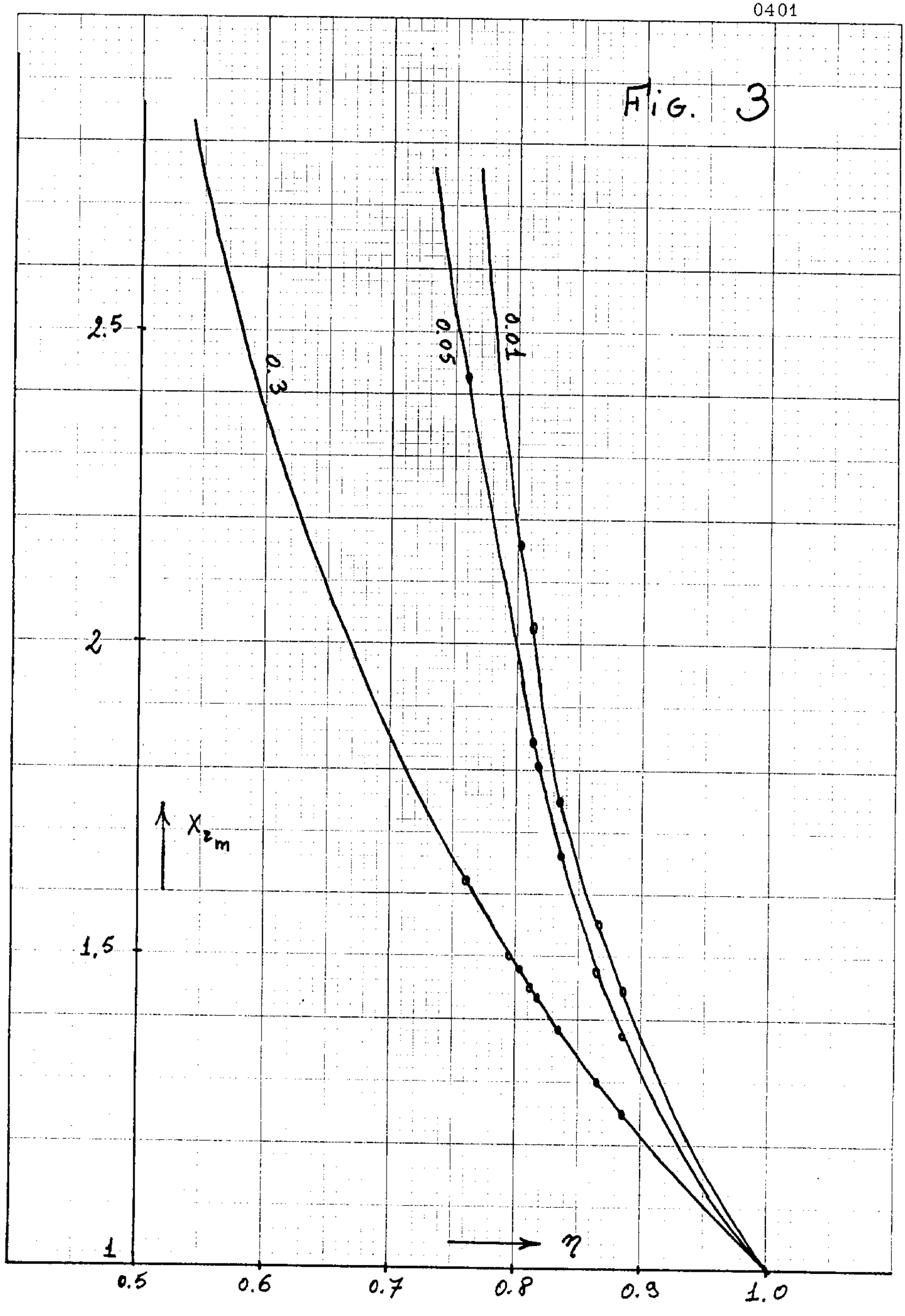
FIG. 3
2.5
2
1,5
1
0.3
0.05
0.01
X_{2m}
η
0.5
0.6
0.7
0.8
0.9
1.0